Partitions of Unity and Vector Fields

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Exercise 1. A vector field X on M is *singular* if $X(p) = 0$ for some $p \in M$, otherwise X is *non-singular*.

1. Find a non-singular smooth vector field on $\mathbb{T}^n$.
2. Find a non-singular smooth vector field on $\mathbb{S}^{2n-1}$. *Hint: identify $\mathbb{S}^{2n-1}$ as a subset of $\mathbb{C}^n$ and use the one parameter subgroup $\varphi^t(z) = (e^{it}z_1, \ldots, e^{it}z_n)$.*
3. Find a non-singular smooth vector field on $\mathbb{RP}^{2n-1}$.
4. Find a smooth vector field in $\mathbb{S}^{2n}$ with one singularity. *Hint: Try using stereographic projections.*

Exercise 2 (Parallelizable manifolds)**.** Let M be a smooth manifold of dimension n. Prove the equivalence of the following properties:

1. M is parallelizable ;
2. There exists vector fields $X_1, \cdots, X_n$, such that $(X_1(p), \cdots, X_n(p))$ is a base of T_pM for all $p \in M$ (equivalently: are independent, or generate T_pM);
3. $\mathcal{X}(M)$ is a free module of dimension $\dim M$ on $\mathcal{C}^\infty(M, \mathbb{R})$.

Exercise 3 (Lie bracket)**.**

1. Let $(x_1, \ldots, x_n)$ be a local system of coordinates on a smooth manifold M, prove that
$$\left[\sum_{i=1}^n X_i \frac{\partial}{\partial x_i}, \sum_{j=1}^n Y_j \frac{\partial}{\partial x_j}\right] = \sum_{i=1}^n \left(\sum_{j=1}^n X_j \frac{\partial Y_i}{\partial x_j} - Y_j \frac{\partial X_i}{\partial x_j}\right) \frac{\partial}{\partial x_i},$$
where X_i's and Y_i's are smooth real maps.
2. Let $\varphi : M \to N$ be a diffeomorphism. Given a vector field X on M, we denote by $\varphi_* X$ the vector field on N defined by $T\varphi \circ X$ (why is it well defined?). Prove that $\varphi_*([X, Y]) = [\varphi_* X, \varphi_* Y]$ for all vector fields X and Y on M.
3. Let $f, g \in \mathcal{C}^\infty(M, \mathbb{R})$, prove that for all vector fields X and Y on M,
$$[fX, gY] = f(X \cdot g)Y - g(Y \cdot f)X + fg[X, Y].$$

Exercise 4 (Plateau function)**.**

1. (a) Find a $\mathcal{C}^\infty$ map f on $\mathbb{R}$ such that $f \geqslant 0$, $f(x) = 0$ as soon as $x \leqslant -1$ or $x \geqslant 0$, and $f(x) > 0$ for $x \in]-1, 0[$.
   (b) Deduce the existence of a non-decreasing $\mathcal{C}^\infty$ map $F : \mathbb{R} \to [0, 1]$, such that $F(x) = 0$ for $x \leqslant -1$, $F(x) = 1$ for $x \geqslant 0$ and $F(x) > 0$ for $x \in]-1, 0[$.
   (c) Now fix $a < b < c < d$. Build a $\mathcal{C}^\infty$ map $P_{a,b,c,d} : \mathbb{R} \to [0, 1]$, such that $P_{a,b,c,d}(x) = 0$ for $x \leqslant a$ or $x \geqslant d$, $P_{a,b,c,d}(x) = 1$ for $x \in [b, c]$, and $P_{a,b,c,d}(x) > 0$ for $x \in]a, b]$ or $x \in [c, d[$.

2. Let M be a manifold and (U, ϕ) be a chart on M such that $\phi(U) = \mathbb{R}^n$. Write $V = \phi^{-1}(B^n(1))$. Prove that there exists a smooth map $f : M \to [0, 1]$ such that $\text{supp}(f) \subset U$ and $f(x) = 1$ for $x \in V$.

Exercise 5 (Extension of a vector field of a submanifold)**.** Let M be a compact submanifold of $\mathbb{R}^n$ and X a smooth vector field on M. Show that there exists a smooth vector field Y on $\mathbb{R}$ such that $X = Y|_M$.

Exercise 6 (Extension of a vector field along a path)**.** Let M be a manifold, $\gamma : [0, 1] \to M$ a smooth path and $X : [0, 1] \to M$ a smooth map such that $X(t) \in T_{\gamma(t)}M$ for all $t \in [0, 1]$. Build a map $\overline{X} : [0, 1] \times M \to TM$ such that $\overline{X}(t, x) \in T_xM$ for all $(t, x) \in [0, 1] \times M \to TM$, and with $\overline{X}(t, \gamma(t)) = X_t$ for all $t \in [0, 1]$.